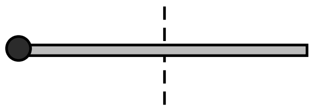
ABRHS PHYSICS (H) NAME: KEY

Moment of Inertia

Note: Feel free to use the chart on the back of the cover page to find moments of inertia.

1. What is the moment of inertia for the rod and mass shown in the diagram. The rod has a mass of 2 kg and a length of 0.5 meters while the little mass on the end of the rod is 0.5 kg. The axis is the center of the rod.



$I = I_{rod,cm} + I_{mass}$

$I = \frac{1}{12}ML^2 + mr^2$; $r = \frac{L}{2}$

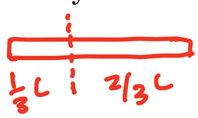
$I = \frac{1}{12}(2)(.5)^2 + (.5)(.25)^2$

$I = 0.042 + .031$

$\boxed{I = .073\ kg \cdot m^2}$

2. What is the moment of inertia for a thin rod (mass M and length L) that is rotated about an axis that is 1/3 of the way from its end? *(There are 3 different ways to do this!)*

① Algebra



$I = I_{left} + I_{right}$

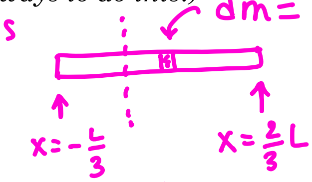
$= \frac{1}{3}\left(\frac{1}{3}M\right)\left(\frac{1}{3}L\right)^2 + \frac{1}{3}\left(\frac{2}{3}M\right)\left(\frac{2}{3}L\right)^2$

$= \frac{1}{81}ML^2 + \frac{8}{81}ML^2 = \frac{9}{81}ML^2$

$\boxed{I = \frac{1}{9}ML^2}$

② Calculus

$dm = \frac{M}{L}dx$



$I = \int r^2 dm = \int_{-L/3}^{2L/3} \frac{M}{L}x^2\,dx = \frac{M}{L}\left(\frac{x^3}{3}\right)\Big|_{-L/3}^{2L/3}$

$= \frac{M}{3L}\left[\left(\frac{2L}{3}\right)^3 - \left(-\frac{L}{3}\right)^3\right] = \frac{M}{3L}\left[\frac{8L^3}{27} + \frac{L^3}{27}\right]$

$= \frac{1}{9}ML^2$:)

3. Imagine a rod and a disk have the same mass and both are being rotated about their center of mass. If the rod has a length of L, what should the radius of the disk be so that they have the same moment of inertia?

$I_{rod} = \frac{1}{12}ML^2$

$I_{disc} = \frac{1}{2}MR^2$

↑ look up in table!

So $\frac{1}{12}ML^2 = \frac{1}{2}MR^2$

$R^2 = \frac{1}{6}L^2$

$\boxed{R = \sqrt{\frac{1}{6}}\,L}$

Moment of Inertia

4. The rotational inertia of a given rod about some axis (perpendicular to the rod) is I. What would happen to the moment of inertia of the rod if
   a. the mass were doubled (but the length stayed the same)?

"I" will be equal to some fraction $\times ML^2$

So doubling mass will double the moment of inertia.

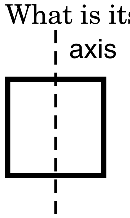
$[\, I \propto M \,]$

   b. the length were doubled (but the mass stayed the same)?

Since the dimensions are squared, $I \propto L^2$

So doubling length will quadruple the moment of inertia

5. Four thin rods are formed into a square. Each rod has a mass of 5 kg and a length of 80 cm. What is its moment of inertia about an axis through its center (as shown)?



$I = I_{\text{horizontal rods}} + I_{\text{vertical rods}}$

$= 2\left[\frac{1}{12}(5)(.8)^2\right] + 2\left[(5)(.4)^2\right]$

all the mass is 40 cm away from rotation axis

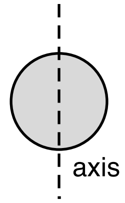
So $I = 2[0.267] + 2[0.8]$

$\boxed{I = 2.13 \text{ kg}\cdot\text{m}^2}$

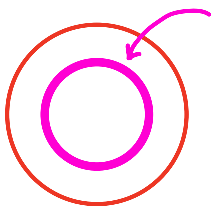
Moment of Inertia

OPTIONAL

6. What is the moment of inertia for a disc of mass M and radius R about its diameter? NOTE: this is an exercise in calculus, so it is optional.



Easiest Way: From the chart of moments of inertia, $I = \frac{1}{2}MR^2$ for a hoop about its diameter. We can find I for a disc about its diameter by adding up a bunch of hoops!

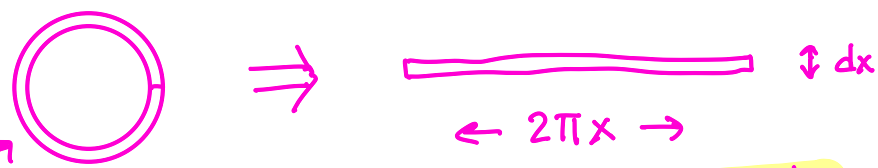


this hoop has a radius of x, and the hoops start with a radius of 0 all the way to a radius of R. Then we we need to figure out the mass of each hoop.

If we call the "density" of the disc $\frac{M}{\pi R^2}$ (total mass/total area)

Then the mass will simply be the density × the area of the hoop. So imagine cutting the hoop, and rolling it out. We get an almost rectangle:*

* in the limit as $dx \to 0$, it is a rectangle!



thickness = dx because it has to be really thin!

Area = $2\pi x\, dx$

So the mass of each hoop, of radius x and thickness dx is

$$m = \frac{M}{\pi R^2}(2\pi x\, dx) = \frac{2M}{R^2} x\, dx$$

So the moment of inertia of each hoop (let's call it "dI" as its a little part of the moment of inertia of the disc) is

$$dI = \frac{1}{2}\left[\frac{2M}{R^2} x\, dx\right] x^2 = \frac{M}{R^2} x^3\, dx$$

Answers:

1) 0.0729 kg m²   *2) 1/9 ML²*   *3)* $\sqrt{\frac{1}{6}L}$ *4. a) x2 b) x4*   *5) 2.13 kg m²*   *6) 1/4 MR²*

OK! Let's put this together:

The moment of inertia of the disc will be the sum of all the moments of inertia of the hoops, so

$$I = \int dI = \int_{x=0}^{x=R} \frac{M}{R^2} x^3 \, dx = \frac{M}{R^2} \frac{x^4}{4} \Big|_0^R$$

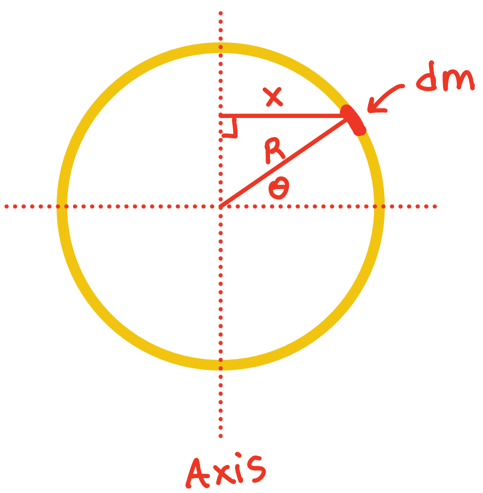
$$I = \frac{M}{R^2}\left[\frac{R^4}{4} - \frac{0^4}{4}\right]$$

$$\boxed{I = \frac{1}{4} M R^2}$$

Notice how easy the actual integral is ;)

This is FYI only! Don't memorize this - it will not show up on any assessment! Only for those that like math!

Let's show where the $I = \frac{1}{2} M R^2$ for the hoop came from! ← Because it is FUN and we are intellectually curious! ;)



The hoop is mass M and radius R.

Let's call the origin the center of the hoop, an rotate it about the y-axis.

Then, we break the hoop into lots of little pieces - "dm", where each piece is a distance x from the rotation axis

Again, the "density" of the hoop would be $\frac{M}{2\pi R}$, since $2\pi R$ would be the circumference of the hoop.

Now, instead of doing this in cartesian coordinates, let's use polar because our hoop only has 1 radius. So the "length" of dm would be $R\,d\theta$ and θ would go all the way around: $\theta: 0 \to 2\pi$. Also notice that $x = R\cos\theta$

Putting this all together:

$$I = \int r^2\,dm = \int_0^{2\pi} \underbrace{\left(\frac{M}{2\pi R}\right)(R\,d\theta)}_{dm}\underbrace{(R\cos\theta)^2}_{x^2}$$

$$\text{So} \quad I = \frac{MR^2}{2\pi}\int_0^{2\pi} \cos^2\theta\, d\theta$$

It looks easy, but unfortunately I don't know how to do that integral. So what does the ~~lazy mathematician~~ physicist do? Look it up! The back of your physics book has a list of integrals, and in it we find

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$$

So let's use that!

$$I = \frac{MR^2}{2\pi}\int_0^{2\pi}(1-\sin^2\theta)\,d\theta = \frac{MR^2}{2\pi}\left[\int_0^{2\pi} d\theta - \int_0^{2\pi}\sin^2\theta\,d\theta\right]$$

$$= \frac{MR^2}{2\pi}\left[\theta - \left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\right]_0^{2\pi}$$

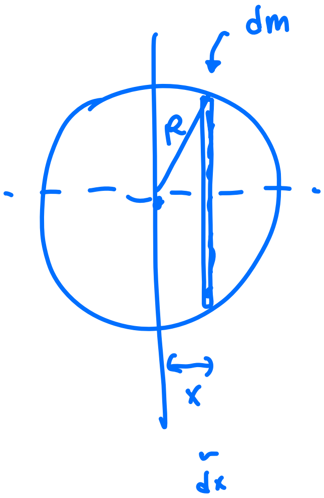
$$= \frac{MR^2}{2\pi}\left[\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}\right]$$

$$= \frac{MR^2}{2\pi}\left[\left(\pi + \underbrace{\cancel{\tfrac{1}{4}\sin(4\pi)}}_{=0}\right) - \left(0 + \underbrace{\cancel{\tfrac{1}{4}\sin(0)}}_{=0}\right)\right]$$

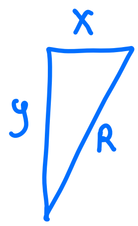
$$= \frac{MR^2}{2\pi}(\pi)$$

$$I = \frac{1}{2}MR^2$$

:)



$$dm = \left(\frac{M}{\pi R^2}\right) 2y\,dx = \frac{M}{\pi R^2}\, 2\sqrt{R^2-x^2}\;dx$$



$$I = \int r^2\,dm$$

$$= \int_{-R}^{R} x^2 \left(\frac{M}{\pi R^2}\, 2\sqrt{R^2-x^2}\;dx\right)$$

$$= \frac{2M}{\pi R^2} \underbrace{\int_{-R}^{R} x^2 \sqrt{R^2-x^2}\;dx}$$

From wolframalpha.com

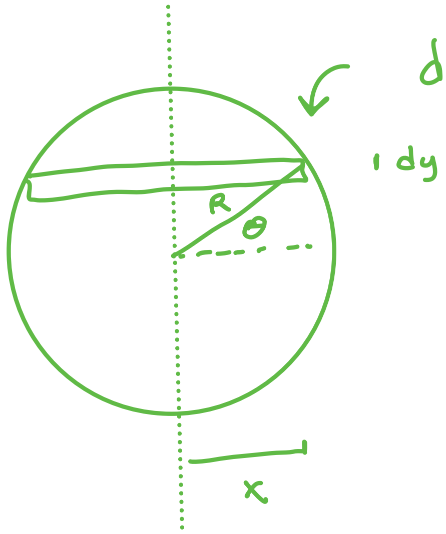
$$= \frac{1}{8}\sqrt{R^2-x^2}\left(-R^2 + \frac{R^3 \sin^{-1}\left(\frac{x}{R}\right)}{\sqrt{1-\frac{x^2}{R^2}}} + 2x^3\right)$$

from limits to $0 \to R$

$$= 2\,\frac{2M}{\pi R^2}\left(\frac{1}{8}\sqrt{R^2-x^2}\left(-R^2 + \frac{R^3 \sin^{-1}\left(\frac{x}{R}\right)}{\sqrt{1-\frac{x^2}{R^2}}} + 2x^3\right)\right)\Bigg|_{\cancel{-R}\;0}^{R}$$

$$= \frac{M}{2\pi R^2}\left[\left(\underbrace{\sqrt{R^2-R^2}}_{=0!}\left(\qquad\right)\right) - \left(\overset{R}{\sqrt{R^2}}\left(-R^2 + \frac{R^3 \overbrace{\sin^{-1}\left(\frac{R}{R}\right)}^{\frac{\pi}{2}}}{\underbrace{\sqrt{1-\frac{R^2}{R^2}}}_{0}} + 2R^3\right)\right)\right.$$

$$= \frac{M}{2\pi R^2}\Bigg($$



$$dI = \frac{1}{12}(dm)(2x)^2$$

$$dm = \frac{M}{\pi R^2}(2x)\,dy$$

$$R^2 = x^2 + y^2 \rightarrow 0 = 2x\,dx + 2y\,dy$$

$$\cancel{y^2 = 2R^2} \qquad x^2 = R^2 - y^2$$

$$\int dI = \int \frac{1}{12}(dm)(2x)^2 = \int \frac{1}{3}(dm)\,x^2$$

$$= \frac{1}{3}\int \underbrace{\left(\frac{M}{\pi R^2}\right)(2x)\,dy}_{dm}\; x^2$$

$$= \frac{2M}{3\pi R^2}\int x^3\,dy$$

$$= \frac{2M}{3\pi R^2}\int \left(\sqrt{R^2 - y^2}\right)^3 dy$$

$$= \frac{2M}{3\pi R^2}\int_{-R}^{R} (R^2 - y^2)^{3/2}\,dy$$